Marco Taboga's Normal Dist. Proofs.

Unknown Mean / Known Variance

$$x = [X_1, \dots, X_n]^T$$

$X_i \overset{iid}{\sim} N(\mu, \sigma^2)$ $\quad \sigma^2$ is known

$$p(X_i \mid \mu) = (2\pi\sigma^2)^{-1/2} \exp\left(-\tfrac{1}{2} \frac{(X_i-\mu)^2}{\sigma^2}\right)$$

via independence:

$$p(x \mid \mu) = \prod_{i=1}^{n} p(X_i \mid \mu) = (2\pi\sigma^2)^{-n/2} \exp\left\{-\tfrac{1}{2\sigma^2} \sum_{i=1}^{n} (X_i-\mu)^2\right)$$

Assume

$$p(\mu) = (2\pi\tau_0^2)^{-1/2} \exp\left\{-\tfrac{1}{2} \frac{(\mu-\mu_0)^2}{\tau_0^2}\right)$$

The posterior is $p(\mu \mid x) = (2\pi\tau_n^2)^{-1/2} \exp\left(-\frac{1}{2\tau_n^2}(\mu-\mu_n)^2\right)$

$$\text{where } \mu_n = \left(\tfrac{n}{\sigma^2} + \tfrac{1}{\tau_0^2}\right)^{-1}\left[\tfrac{n}{\sigma^2}\left(\tfrac{1}{n}\sum X_i\right) + \tfrac{1}{\tau_0^2}\mu_0\right]$$

$$\text{\& } \tau_n^2 = \left(\tfrac{n}{\sigma^2} + \tfrac{1}{\tau_0^2}\right)^{-1}$$

Proof.

$$\begin{aligned}
p(\mu \mid x) &= p(\mu)\, p(x \mid \mu) \\
&= (2\pi\tau_0^2)^{-1/2} \exp\left\{-\tfrac{1}{2}\frac{(\mu-\mu_0)^2}{\tau_0^2}\right\} \cdot (2\pi\sigma^2)^{-n/2} \exp\left\{-\tfrac{1}{2\sigma^2}\sum (X_i-\mu)^2\right\} \\
&= (2\pi\sigma^2)^{-n/2}(2\pi\tau_0^2)^{-1/2} \exp\left(-\tfrac{1}{2}\underbrace{\left[\tfrac{1}{\sigma^2}\sum (X_i-\mu)^2 + \tfrac{1}{\tau_0^2}(\mu-\mu_0)^2\right]}_{=q}\right) \\
&= (2\pi\sigma^2)^{-n/2}(2\pi\tau_0^2)^{-1/2} \exp\left\{-\tfrac{1}{2} q\right\}
\end{aligned}$$

Note

$$\begin{aligned}
\sum_{i=1}^{n} (X_i-\mu)^2 &= \sum_{i=1}^{n} (X_i - \bar{X} + \bar{X} - \mu)^2 \\
&= \sum_{i=1}^{n} (X_i-\bar{X})^2 + 2(\bar{X}-\mu)\sum_{i=1}^{n}(X_i-\bar{X}) + n(\bar{X}-\mu)^2 \\
&= \sum_{i=1}^{n} (X_i-\bar{X})^2 + n(\bar{X}-\mu)^2
\end{aligned}$$

$$q = \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 + \frac{1}{\tau_0^2}(\mu-\mu_0)^2$$

Using the above:

$$= \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\bar{x})^2 + \frac{n}{\sigma^2}\underbrace{(\bar{x}-\mu)^2}_{\text{expand}} + \frac{1}{\tau_0^2}\underbrace{(\mu-\mu_0)^2}_{\text{expand}}$$

$$= \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\bar{x})^2 + \frac{n}{\sigma^2}\bar{x}^2 - 2\frac{n}{\sigma^2}\bar{x}\mu + \frac{n}{\sigma^2}\mu^2 + \frac{1}{\tau_0^2}\mu^2 + \frac{1}{\tau_0^2}\mu_0^2 - 2\frac{1}{\tau_0^2}\mu_0\mu$$

$$= \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\bar{x})^2 + \frac{n}{\sigma^2}\bar{x}^2 + \frac{1}{\tau_0^2}\mu_0^2 + \underbrace{\left(\frac{n}{\sigma^2}+\frac{1}{\tau_0^2}\right)}_{(\tau_n^2)^{-1}}\mu^2 - 2\left(\frac{n}{\sigma^2}\bar{x}+\frac{1}{\tau_0^2}\mu_0\right)\mu$$

$$= \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\bar{x})^2 + \frac{n}{\sigma^2}\bar{x}^2 + \frac{1}{\tau_0^2}\mu_0^2 - \frac{1}{\tau_n^2}\mu_n^2 + \frac{1}{\tau_n^2}(\mu^2 - 2\mu_n\mu + \mu_n^2)$$

$$\tau_n^2 = \left(\frac{n}{\sigma^2}+\frac{1}{\tau_0^2}\right)^{-1}$$

$$\mu_n = \left(\frac{n}{\sigma^2}+\frac{1}{\tau_0^2}\right)^{-1}\left[\frac{n}{\sigma^2}\left(\frac{1}{n}\sum x_i\right) + \frac{1}{\tau_0^2}\mu_0\right]$$

$$\therefore\ q = \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\bar{x})^2 + \frac{n}{\sigma^2}\bar{x}^2 + \frac{1}{\tau_0^2}\mu_0^2 - \frac{1}{\tau_n^2}\mu_n^2 + \frac{1}{\tau_n^2}(\mu-\mu_n)^2$$

$$p(x,\mu) = (2\pi\sigma^2)^{-n/2}(2\pi\tau_0^2)^{-1/2}\exp\{-\tfrac{1}{2}q\}$$

$$= (2\pi\sigma^2)^{-n/2}(2\pi\tau_0^2)^{-1/2}\exp\left\{-\tfrac{1}{2}\left[\frac{1}{\sigma^2}\sum(x_i-\bar{x})^2 + \frac{n}{\sigma^2}\bar{x}^2 + \frac{1}{\tau_0^2}\mu_0^2 - \frac{1}{\tau_n^2}\mu_n^2\right]\right\}$$
$$\cdot \exp\left\{-\frac{1}{2\tau_n^2}(\mu-\mu_n)^2\right\}$$

$$= (2\pi\sigma^2)^{-n/2}(2\pi\tau_0^2)^{-1/2}\exp\left\{-\frac{1}{2\sigma^2}\left[\sum(x_i-\bar{x})^2 + n\bar{x}^2 + \frac{\sigma^2}{\tau_0^2}\mu_0^2 - \frac{\sigma^2}{\tau_n^2}\mu_n^2\right]\right\}$$
$$\cdot (2\pi\tau_n^2)^{-1/2}\exp\left(-\frac{1}{2\tau_n^2}(\mu-\mu_n)^2\right)$$

$$= \underbrace{h(x)}_{\text{pdf dependent on } x_i \text{ but not on}}\ g(\mu, x)$$

pdf that depends on x thru μ_n g is $N(\mu_n, \tau_n^2)$

Via factorization we obtain

$$p(\mu|x) = g(\mu, x)$$
$$p(x) = h(x)$$

Therefore $\mu|x \sim N(\mu_n, \tau_n^2)$